Quaternion Vortex Methods

Leonardo Traversoni
Uruguay Foreign Ministry, Uruguay
ltraversoni@hotmail.com

QUATERNION VORTEX METHODS

We show the advantages of using the quaternionic expression of some PDEs in this case Navier Stokes in this case to obtain a faster and more simple numerical solution via vortex methods.

The Navier Stokes equations for incompressible viscous flow are:

$$\frac{Du}{Dt} = -\bigtriangledown P + \frac{1}{R}\bigtriangledown^2 u \ \ in\, D \tag{1}$$

$$\bigtriangledown \cdot u = 0 \ \ in\, D \tag{2}$$

$$u = 0 \ \ on\, \partial D \tag{3}$$

Taking ξ as:

$$\xi = \bigtriangledown \times v \tag{4}$$

The vorticity we have the vorticity transport equation:

$$\frac{D\xi}{Dt} = (\xi \cdot \bigtriangledown)u + \frac{1}{R}\bigtriangledown^2 \xi \tag{5}$$

Here u is the velocity, P the pressure, R the Reynolds number.

As $\bigtriangledown \cdot u = 0$ and $\xi = \bigtriangledown \times u$

there exists a vector function $\psi(x)$ such that $u = \bigtriangledown \times \psi$ then

$$\bigtriangledown^2 \psi = -\xi \tag{6}$$

In 3D ψ is the velocity potencial in 2D is the stream function.

Applying ideas of quaternionic and Clifford analysis we can find a transformation into one non-linear equation only for the vorticity ξ. For this reason we use the higher-dimensional version of the Borel-Pompeiu formula:

$$TDu(x) = u(x) - Fu(x) \tag{7}$$

where T is the T-operator (Teodorescu transform), D the Dirac-operator and F the Cauchy integral. The Cauchy integral depends only on the boundary values of u.

That means that if $u = 0$ on the boundary then this part can be deleted of the formula.

Moreover, Du means for a quaternion valued function $(0, u)$ (u is the vector of velocity)

$$Du = (-div\, u,\ rot\, u) \tag{8}$$

As we are working with divergence free vectors and consequently

$$Du = (0,\ rotu)$$

Remembering that

$$rot\ u = \nabla \times u$$

we have that

$u = TDu$ and with $Du = rot\, u = \xi$ it follows

$$u = T\xi \tag{9}$$

This is an expression to describe the velocity u explicitely by the vorticity ξ. If the boundary values of u are not zero but some known quantity then we have this additional known summand F(boundary values of u). The operators T and F are defined as:

$$(T_G u)(x) = -\int_G e(x-y)u(y)dG_y$$

$$(F_\gamma u)(x) = \int_\gamma e(x-y)\alpha(y)u(y)d\gamma_y$$

α is the outer normal to γ at the point y and $e(x)$ the fundamental solution (generalized Cauchy kernel) of the Dirac-operator.

In this way substituting in the above equations we obtain a nonlinear equation in ξ instead a system in u and ξ. To find representation formulas and numerical methods for ξ is one of the goals of the project. Because we have to evaluate only the vorticity (and not in addition the velocity, too) a better efficiency of this approach is expected.

So to apply numerical methods to the equations we only need to have the numerical expressions of the main operators: We show the discrete expression of the main operators used:

$$\Delta u = \sum D^{+}\_k, jD^{-}\_k, ju \tag{10}$$

$$D^{+}D^{-} = \frac{1}{h}\left[u_{k+1,j} - u_{j,k} - u_{k+1,j+1} + u_{k,j-1}\right] \tag{11}$$

The Teodorescu has also an expression:

$$(T\_hf) = \sum \_y = G\_ke\_h(x-y)f(y)h^3 \tag{12}$$

Where e is the fundamental solution of D We also have that:

$$(F\_hf)(x) = \sum \_y = G\_ke\_h(x-y)\alpha(y)f(y)h^2 \tag{13}$$

So now we only have to apply the above in the classical vortex method as follows: 1) Create an initial particle field that approximates ξ_i , by placing uniformly spaced particles into the support of ξ_i and by setting their vectorial circulation to the local value of ξ_i

2) Create an triangulation $S = S_i$ of the boundaries and place immovable vortex particles on the triangles' centres. 3) Use a standard time-stepping technique, e.g., a Runge-Kutta method or a multistep method, for advancing the ODEs in time. In order to evaluate the vorticities and their gradients in each (sub-)step, do the following: ξ_i at each particle location as well as at the quadrature points on the boundaries with the help of the Biot-Savart law b) Compute

$\nabla\xi_i$

at each particle location, again by using the Biot-Savart law. c) Compute the unknown vortex sheet strength on the boundaries. d) Now pretend the particles on the boundaries are ordinary particles. Use the Biot- Savart law (6) in order to compute ξ_i

and $\nabla\xi_i$

at each particle inside the domain. 4. Remove any particles that might have escaped the fluid domain. This should only rarely happen if the surface triangulation is sufficiently refined. 5. Repeat steps 3 and 4 until the requested termination time is reached