Collocation method for solving singular ODEs and higher index DAEs

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During recent years, a lot of scientific work concentrated on the analysis and numerical treatment of boundary value problems (BVP) in ordinary differential equations (ODEs) which can exhibit singularities. Such problems have often the following form:

$$y'(t) = \frac{1}{t^\alpha} M(t)y(t) + f(t, y(t)), \quad t \in (0, 1], \quad b(y(0), y(1)) = 0.$$

For $\alpha = 1$ the problem is called singular with a singularity of the first kind, for $\alpha > 1$ it is essentially singular (singularity of the second kind).

The search for efficient numerical methods to solve the above BVP is strongly motivated by numerous applications from physics, chemistry, mechanics, ecology, or economy. In particular, problems posed on infinite intervals are frequently transformed to a finite domain taking above form with $\alpha > 1$. Also, research activities in related fields, like differential algebraic equations (DAEs) or singular Sturm-Liouville eigenvalue problems benefit from techniques developed for singular BVPs.

While collocation stays robust and shows advantageous convergence properties in context of singular problems, other high order methods suffer from instabilities and order reductions. It turns out that for singular BVPs with smooth solutions, the convergence order of the polynomial collocation is at least equal to the so-called stage order of the method. For collocation at equidistant points or Gaussian points this convergence result means that the scheme with m inner collocation points constitutes a high order basic solver whose global error is $O(h^m)$ uniformly in t [5].

Clearly, in order to solve an ODE system efficiently, the error estimate and the mesh adaptation strategy have to be provided to correctly reflect the solution behavior. Due to the robustness of collocation, this method was used in one of the best established standard FORTRAN codes for (regular) BVPs, COLSYS [1], as well as in Matlab codes bvp4c [7], the standard module for (regular) ODEs with an option for singular problems, BVP SOLVER [8], sbvp [2], and bvpsuite [6]. The scope of bvpsuite includes fully implicit form of the ODE system with multi-point boundary conditions, arbitrary mixed order of the differential equations including zero, module for dealing with infinite intervals, module for eigenvalue problems, free parameters, and a path-following strategy for parameter-dependent problems with turning points.

We will illustrate how bvpsuite can be used to solve BVPs from applications, Complex Ginzburg-Landau equations, density profile in hydrodynamics, and generalized Korteweg-de Vries equation [3].

Finally, we turn to higher index DAEs. Higher index DAEs constitute a really challenging class of problems due to the involved differentiation which is a critical operation to carry out numerically. A possible technique to master the problem is to pre-handle the DAE system in such a way that the transformed problem is of index one and less difficult to solve. Since this approach is technically involved, it is worth to try to avoid it and provide a method which can be applied directly to the original DAE system of high index. At present, there are only some experimental results available, but they are quite encouraging and therefore, we shall briefly discuss them during the presentation [4].

References

[1] U. Ascher, J. Christiansen, and R. Russell, Collocation software for boundary value ODEs. ACM Transactions on Mathematical Software, 1981, pp. 209–222.

[2] W. Auzinger, G. Kneisl, O. Koch, and E.B. Weinmüller, A collocation code for boundary value problems in ordinary differential equations. Numer. Algorithms, 2003, pp. 27–39. Code available from http://www.mathworks.com/matlabcentral/fileexchange > Mathematics > Differential Equations > SBVP1.0 Package.

[3] C. Budd, O. Koch, S. Schirnhofer, and E.B. Weinmüller, Work in progress, 2014.
[4] M. Hanke, R. März, C. Tischendorf, E.B. Weinmüller, and S. Wurm, Work in progress, 2014.
[5] F. de Hoog and R. Weiss, Collocation Methods for Singular BVPs. SIAM J. Numer. Anal., 1978, pp. 198–217.
[6] G. Kitzhofer, O. Koch, G. Pulverer, C. Simon, and E.B. Weinmüller, Numerical Treatment of Singular BVPs: The New Matlab Code bvpsuite. JNAIAM J. Numer. Anal. Indust. Appl. Math., 2010, pp. 113–134.
[7] L. Shampine, J. Kierzenka, and M. Reichelt, Solving Boundary Value Problems for Ordinary Differential Equations in Matlab with bvp4c. 2000.
[8] L. Shampine, P. Muir, and H. Xu, A User-Friendly Fortran BVP Solver. JNAIAM J. Numer. Anal. Indust. Appl. Math., 2006, pp. 201–217.