Some results on invariant measures of reduced discrete mechanical systems

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The interest in discrete (time) mechanical systems is highly motivated by the construction of structure preserving numerical integrators for the continuous ones [MW01]. For instance, one of the features of their discrete evolution consists in the invariance of a Liouville measure, which is associated to an invariant canonical symplectic form on the discrete phase space; the latter is formed by pairs of positions in the configuration space (a differentiable manifold).

Another known fact about discrete mechanical systems is the momentum preservation in the presence of symmetries. Moreover, there is a well developed discrete counterpart of reduction theory of continuous systems. When passing to a dynamical system on a quotient space in order to remove the symmetries, it is worth saying that the latter result in hamiltonian systems on a Poisson manifold. In this sense, they preserve a Poisson bracket but not necessarily a symplectic form, and, thus, it is not guaranteed the existence of an invariant measure as in the previous case; the situation is analogue for discrete systems.

When satisfied, the property of unimodularity of a Poisson manifold gives an affirmative answer to the question of whether such a reduced continuous system preserves a measure. This is due to the equivalence between the notion of unimodularity and the existence of a volume form that is invariant by all hamiltonian flows [Wei97,FGM13].

It is the purpose of this work to study sufficient conditions to relate the concept of unimodulatiry of a Poisson manifold to the existence of invariant measures of reduced discrete mechanical systems. To this end, we firstly address the problem for the discrete Euler-Poincaré equations under quite general conditions. These equations describe the reduced dynamics of discrete systems whose configuration space, G, is simultaneously a group of symmetries acting on the discrete phase space, $G\times G$, by the diagonal action induced by the left multiplication [BS99]. Reinterpreting the idea of unimodularity as the unimodularity of $(G \times G)/G \cong G$ as a Lie group [Koz88], we use results from Lie algebra theory to show that this is sufficient to prove the invariance of a certain measure.

Finally, towards a more general result, we tackle the problem of existence of invariant measures of reduced discrete mechanical systems by means of geometric arguments, instead of dealing with their equations of motion. These take into account the presevation of the reduced Poisson structure on the reduced discrete space and, also, the symplectic leaves associated to it. Although this work-in-progress technique requires stronger hypotheses when it is applied and compared to the previous particular situation, it is intended to cover a wider range of systems. As future work, our next step will be to extend the current approach to include (nonholonomic) constraints.

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