Schubert varieties and distances between subspaces of different dimensions

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We resolve two fundamental problems regarding subspace distances that have arisen considerably often in applications: How could one define a notion of distance between (i) two linear subspaces of different dimensions, or (ii) two affine subspaces of the same dimension, in a way that generalizes the usual Grassmann distance between equidimensional linear subspaces? We show that (i) is the distance of a point to a Schubert variety, and (ii) is the distance within the Grassmannian of affine subspaces. In our context, a Schubert variety and the Grassmannian of affine subspaces are both regarded as subsets of the usual Grassmannian of linear subspaces. Combining (i) and (ii) yields a notion of distance between (iii) two affine subspaces of different dimensions. Aside from reducing to the usual Grassmann distance when the subspaces in (i) are equidimensional or when the affine subspaces in (ii) are linear subspaces, these distances are intrinsic and do not depend on any embedding of the Grassmannian into a larger ambient space. Furthermore, they can all be written down as concrete expressions involving principal angles, and are efficiently computable in numerically stable ways. We show that our results are largely independent of the Grassmann distance — if desired, it may be substituted by any other common distances between subspaces. Central to our approach to these problems is a concrete algebraic geometric view of the Grassmannian that parallels the differential geometric perspective that is now well-established in applied and computational mathematics. A secondary goal of this article is to demonstrate that the basic algebraic geometry of Grassmannian can be just as accessible and useful to practitioners.

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